Plastic Spin, Gradients/Localization & Texture
[From Micro/Single Slip to Macro/Phenomenological Plasticity]

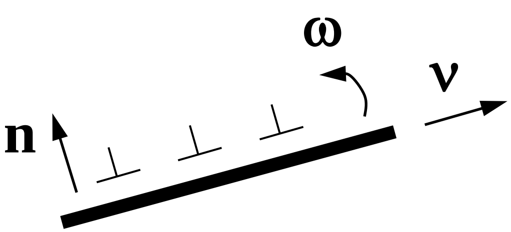
■ A Scale Invariance Argument

- ***Momentum Balance for Dislocated State***

$$\mathrm{div}\mathbf{T}^{\mathbf{D}} = \hat{\mathbf{f}} \; ; \quad \mathbf{T}^{\mathbf{D}} = \mathbf{S} - \mathbf{T}^{\mathrm{L}} \; ; \quad \mathrm{div}\mathbf{S} = 0$$

$\mathbf{T}^{\mathbf{D}}$...dislocation stress; $\hat{\mathbf{f}}$...dislocastion-lattice interaction force

- ***Yield Condition:*** $\mathrm{f} = \hat{\mathbf{f}} \cdot \boldsymbol{\nu} = 0; \quad \hat{\mathbf{f}} = \left(\hat{\alpha} + \hat{\beta}\mathrm{j} - \hat{\gamma}\tau^{\mathrm{L}}\right)\boldsymbol{\nu}, \quad \tau^{\mathrm{L}} = \mathbf{T}^{\mathbf{L}} \cdot \mathbf{M}$



$$\mathbf{M} = \left(\boldsymbol{\nu} \otimes \mathbf{n}\right)_{\mathrm{s}}, \quad \boldsymbol{\Omega} = \left(\boldsymbol{\nu} \otimes \mathbf{n}\right)_{\mathrm{a}}, \quad \dot{\boldsymbol{\nu}} = \boldsymbol{\omega}\boldsymbol{\nu}$$

$$\mathbf{D}^{\mathbf{p}} = \dot{\gamma}^{\mathrm{p}}\mathbf{M}, \quad \mathbf{W}^{\mathbf{p}} = \dot{\gamma}^{\mathrm{p}}\boldsymbol{\Omega}, \quad \mathbf{T}^{\mathbf{D}} = \mathrm{t}_{\mathrm{m}}\mathbf{M} + \mathrm{t}_{\mathrm{n}}\mathbf{N}$$

$$\max\left\{\mathrm{tr}\mathbf{T}^{\mathbf{L}}\mathbf{D}^{\mathbf{p}}\right\}; \;\; \mathrm{tr}\mathbf{M} = 0, \;\; \mathrm{tr}\mathbf{M}^2 = 1/2 \quad \Rightarrow \quad \mathbf{D}^{\mathbf{p}} = \frac{\dot{\gamma}^{\mathrm{p}}}{2\sqrt{\mathrm{J}}}\mathbf{T}^{\mathbf{L}\prime}; \quad \mathrm{J} = \frac{1}{2}\mathrm{tr}\left(\mathbf{T}^{\mathbf{L}\prime}\mathbf{T}^{\mathbf{L}\prime}\right)$$

$$\therefore \quad \tau = \sqrt{\mathrm{J}} = \kappa\left(\gamma^{\mathrm{p}}\right)$$

• *Structure of Macroscopic Anisotropic Hardening Plasticity*

$$\mathbf{D}^{\mathbf{p}} = \frac{\dot{\gamma}^{p}}{2\sqrt{J}}\left(\boldsymbol{\sigma}' - \boldsymbol{\alpha}'\right); \qquad \overset{\circ}{\boldsymbol{\alpha}} = \left(\frac{\dot{t}_m}{\dot{\gamma}^{p}} - \frac{\dot{t}_n t_m}{t_n \dot{\gamma}^{p}}\right)\mathbf{D}^{\mathbf{p}} + \frac{\dot{t}_n}{t_n}\boldsymbol{\alpha}$$

$$\overset{\circ}{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}} - \boldsymbol{\omega}\boldsymbol{\alpha} + \boldsymbol{\alpha}\boldsymbol{\omega}; \qquad \boldsymbol{\omega} = \mathbf{W} - \mathbf{W}^{\mathbf{p}}, \qquad \mathbf{W}^{\mathbf{p}} = -\frac{1}{t_n}\left(\boldsymbol{\alpha}\mathbf{D}^{\mathbf{p}} - \mathbf{D}^{\mathbf{p}}\boldsymbol{\alpha}\right)$$

$$\dot{\gamma}^{p} = \frac{\overset{\circ}{\boldsymbol{\sigma}}' \cdot \left(\boldsymbol{\sigma}' - \boldsymbol{\alpha}'\right)}{\kappa\left(t_m' + 2\kappa'\right)}; \qquad \begin{cases} \dot{f} = 0 \text{ ...consistency condition} \\ f = \frac{1}{2}\left(\boldsymbol{\sigma}' - \boldsymbol{\alpha}'\right)\cdot\left(\boldsymbol{\sigma}' - \boldsymbol{\alpha}'\right) - \kappa^{2} = 0 \text{ ...yield condition} \end{cases}$$

• *Example Models*

M1: $\dot{t}_m = c\dot{\gamma}^{p}; \quad \dot{t}_n = 0 \rightarrow \begin{cases} \overset{\circ}{\boldsymbol{\alpha}} = c\mathbf{D}^{\mathbf{p}} \\ \mathbf{W}^{\mathbf{p}} = \zeta\left(\boldsymbol{\alpha}\mathbf{D}^{\mathbf{p}} - \mathbf{D}^{\mathbf{p}}\boldsymbol{\alpha}\right) \end{cases}$ … Prager

M2: $\left.\begin{aligned} \dot{t}_m &= c\dot{\gamma}^{p} - \bar{c}\dot{\gamma}^{p} t_m \\ \dot{t}_n &= -\bar{c}\dot{\gamma}^{p} t_n \end{aligned}\right\} \rightarrow \begin{cases} \overset{\circ}{\boldsymbol{\alpha}} = c\mathbf{D}^{\mathbf{p}} - \bar{c}\dot{\gamma}^{p}\boldsymbol{\alpha} \\ \mathbf{W}^{\mathbf{p}} = \zeta e^{\bar{c}\gamma^{p}}\left(\boldsymbol{\alpha}\mathbf{D}^{\mathbf{p}} - \mathbf{D}^{\mathbf{p}}\boldsymbol{\alpha}\right) \end{cases}$ … Armstrong-Frederick

- ***Inhomogeneous Back Stress:*** $\boldsymbol{T^D = \alpha + T^{inh}}$
  - $\boldsymbol{\alpha}$ = *homogeneous back stress … as before*

$$\mathbf{T}^{\mathbf{inh}} = \hat{\mathbf{g}}\left(\mathbf{n}, \boldsymbol{\nu}, \nabla\gamma^{p}\right)$$

$$\approx \left[\mathbf{n} \otimes \nabla\gamma^{p} + \left(\nabla\gamma^{p}\right) \otimes \mathbf{n}\right] + \left[\boldsymbol{\nu} \otimes \nabla\gamma^{p} + \left(\nabla\gamma^{p}\right) \otimes \boldsymbol{\nu}\right]$$

$$\mathrm{div}\mathbf{T}^{\mathbf{inh}} \approx \left(\mathbf{n} + \boldsymbol{\nu}\right)\nabla^{2}\gamma^{p} + \left(\mathbf{grad}^{2}\gamma^{p}\right)\left(\mathbf{n} + \boldsymbol{\nu}\right)$$

$$\left(\mathrm{div}\mathbf{T}^{\mathbf{inh}}\right) \cdot \boldsymbol{\nu} \approx \nabla^{2}\gamma^{p} + \gamma^{p}_{,ij}\left(\nu_{i}\nu_{j} + \nu_{i}n_{j}\right)$$

  - *Integrate over all possible orientations of* $(\boldsymbol{n}, \boldsymbol{\nu})$

$$\left(\mathrm{div}\mathbf{T}^{\mathbf{inh}}\right) \cdot \boldsymbol{\nu} \rightarrow \nabla^{2}\gamma^{p}$$

$$\therefore \quad \tau = \kappa\left(\gamma^{p}\right) - c\nabla^{2}\gamma^{p}$$

- ***Same Procedure for Nanopolycrystals***
  - *Representative slip plane → Representative planar GB*

■ A Note on Consistency with Continuum Thermodynamics

Thermodynamics applied to gradient theories :
The theories of Aifantis and Fleck & Hutchinson and their generalization
[*J. Mech. Phys. Sol.* **57**, 405-421 (2009)]

M.E. Gurtin/Carnegie-Mellon & L. Anand/MIT

Abstract : We discuss the physical nature of flow rules for rate-independent (gradient) plasticity laid down by Aifantis and Fleck and Hutchinson. As central results we show that:

- the flow rule of Fleck and Hutchinson is incompatible with thermodynamics unless its nonlocal term is dropped.
- If the underlying theory is augmented by a general defect energy dependent on γ^p and $\nabla\gamma^p$, then compatibility with thermodynamics requires that its flow rule reduce to that of Aifantis.

Refs

• E.C. Aifantis, On the microstructural origin of certain inelastic models, *Trans. ASME, J. Engng. Mat. Tech.* **106**, 326-330 (1984).

• E.C. Aifantis, The physics of plastic deformation, *Int. J. Plasticity* **3**, 211-247 (1987).

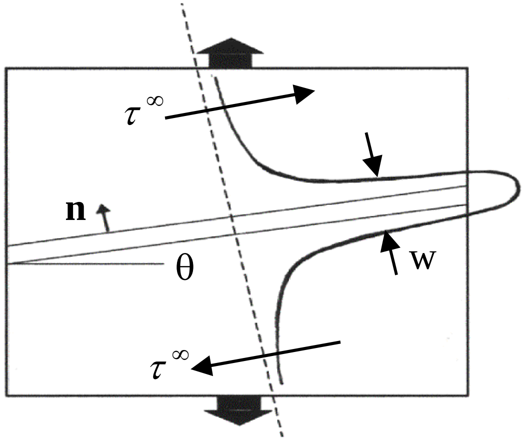
• N.A. Fleck and J.W. Hutchinson, A reformulation of strain gradient plasticity, *J. Mech. Phys. Solids* **49**, 2245-2271 (2001).

■ A Note on Shear Band Widths/Spacings

- ***Constitutive Eq.***

$$\mathbf{S}' = -p\mathbf{1} + 2\mu\mathbf{D} \quad ;$$

$$\mu = \frac{\tau}{\dot{\gamma}} \quad , \quad \begin{cases} \tau \equiv \sqrt{\tfrac{1}{2}\mathbf{S}'\cdot\mathbf{S}'} \\ \dot{\gamma} \equiv \sqrt{2\mathbf{D}\cdot\mathbf{D}} \end{cases} ; \quad \tau = \kappa(\gamma) - c\nabla^2\gamma$$

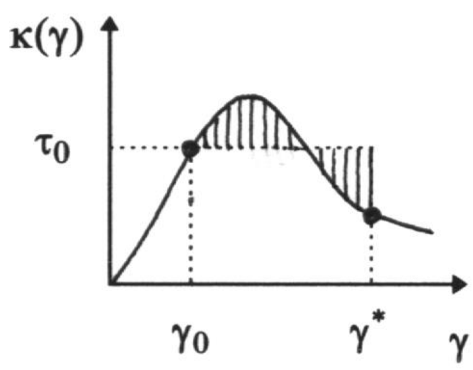


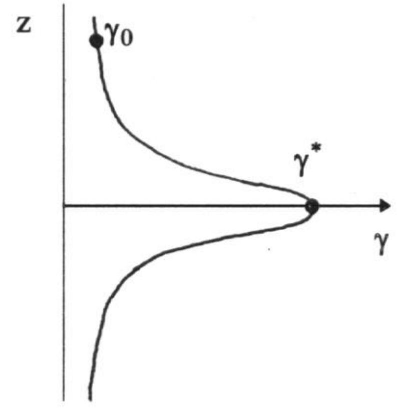
- ***Linear Stability / SB Orientation***

$$\mathrm{v} = \mathrm{L}_\infty \mathrm{x} + \tilde{\mathrm{v}}\mathrm{e}^{\mathrm{i}qz+\omega t} ; \quad \omega > 0 \quad (\&\omega_{\max}) \;\rightarrow\; \theta_{cr} = \frac{\pi}{4} \quad \& \quad \begin{cases} h_{cr} = 0 \\ q_{cr} = 0 \end{cases}$$

- ***Nonlinear Solution / SB Thickness***

$$c\gamma_{zz} = \kappa(\gamma) - \tau^\infty$$

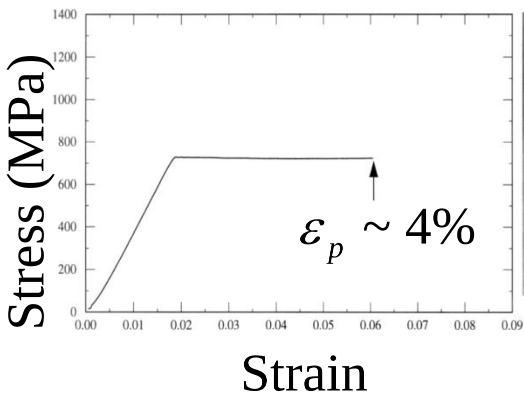


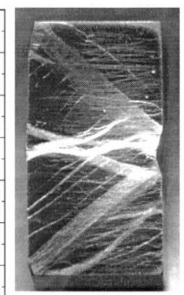


• *Multiple SBs in Bulk Nanostructured Fe-10% Cu Polycrystals*

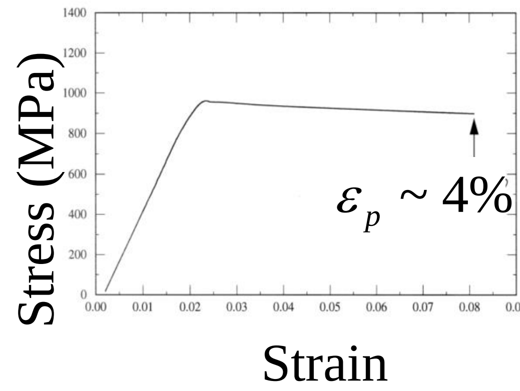
- *Compression tests*

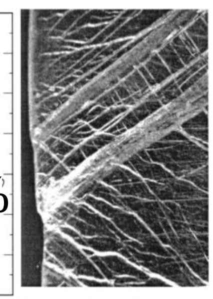
d ~1370 nm, σ_y ~750 Mpa
angle ~ 49^0





d ~540 nm, σ_y ~960 MPa
angle ~ 49^0





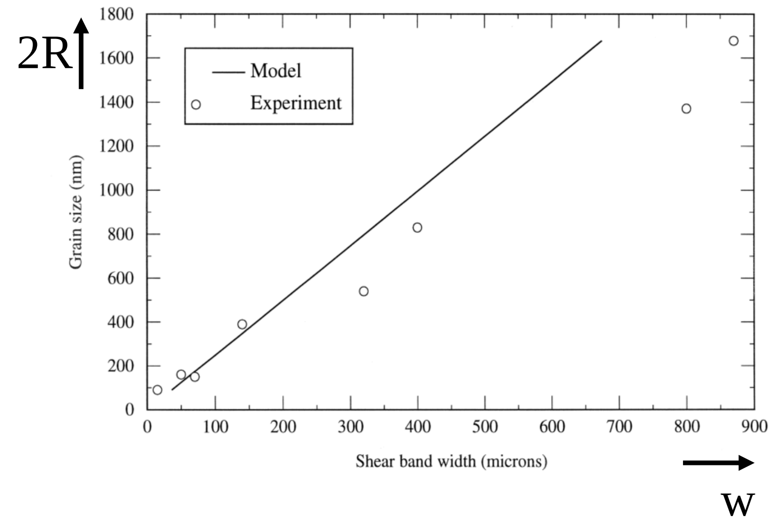
- *Shear band width analysis*

$$\tau = \kappa(\gamma) - c\nabla^2\gamma$$

$$w \sim 0.4\sqrt{c}$$

$$c \sim \frac{R^2}{10}(\beta + h)$$

$$\beta = \alpha G \frac{7 - 5\nu}{15(1-\nu)}$$



■ A Note on Rotational Bands

Simple shear (no elastic deformation)

$$[\boldsymbol{D}]=[\boldsymbol{D}^{\boldsymbol{p}}]=\frac{\dot{\gamma}^{p}}{2}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\qquad [\boldsymbol{W}]=\frac{\dot{\gamma}^{p}}{2}\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}$$

Armstrong-Frederick hardening rule

$$\dot{a}_m=c_1\dot{\gamma}^p-c_2\dot{\gamma}^p\alpha_m\,,\qquad \dot{a}_n=-c_2\dot{\gamma}^p\alpha_n$$

$$\Rightarrow\begin{cases}\mathring{\boldsymbol{\alpha}}=c_1\boldsymbol{D}^{\boldsymbol{p}}-c_2\dot{\gamma}^p\boldsymbol{\alpha}\,; & \mathring{\boldsymbol{\alpha}}=\dot{\boldsymbol{\alpha}}-\boldsymbol{\omega\alpha}+\boldsymbol{\alpha\omega}\\ \boldsymbol{W}^{\boldsymbol{p}}=\zeta e^{c_2\gamma^p}\left(\boldsymbol{\alpha D}^{\boldsymbol{p}}-\boldsymbol{D}^{\boldsymbol{p}}\boldsymbol{\alpha}\right); & \boldsymbol{\omega}=\boldsymbol{W}-\boldsymbol{W}^{\boldsymbol{p}}\end{cases}$$

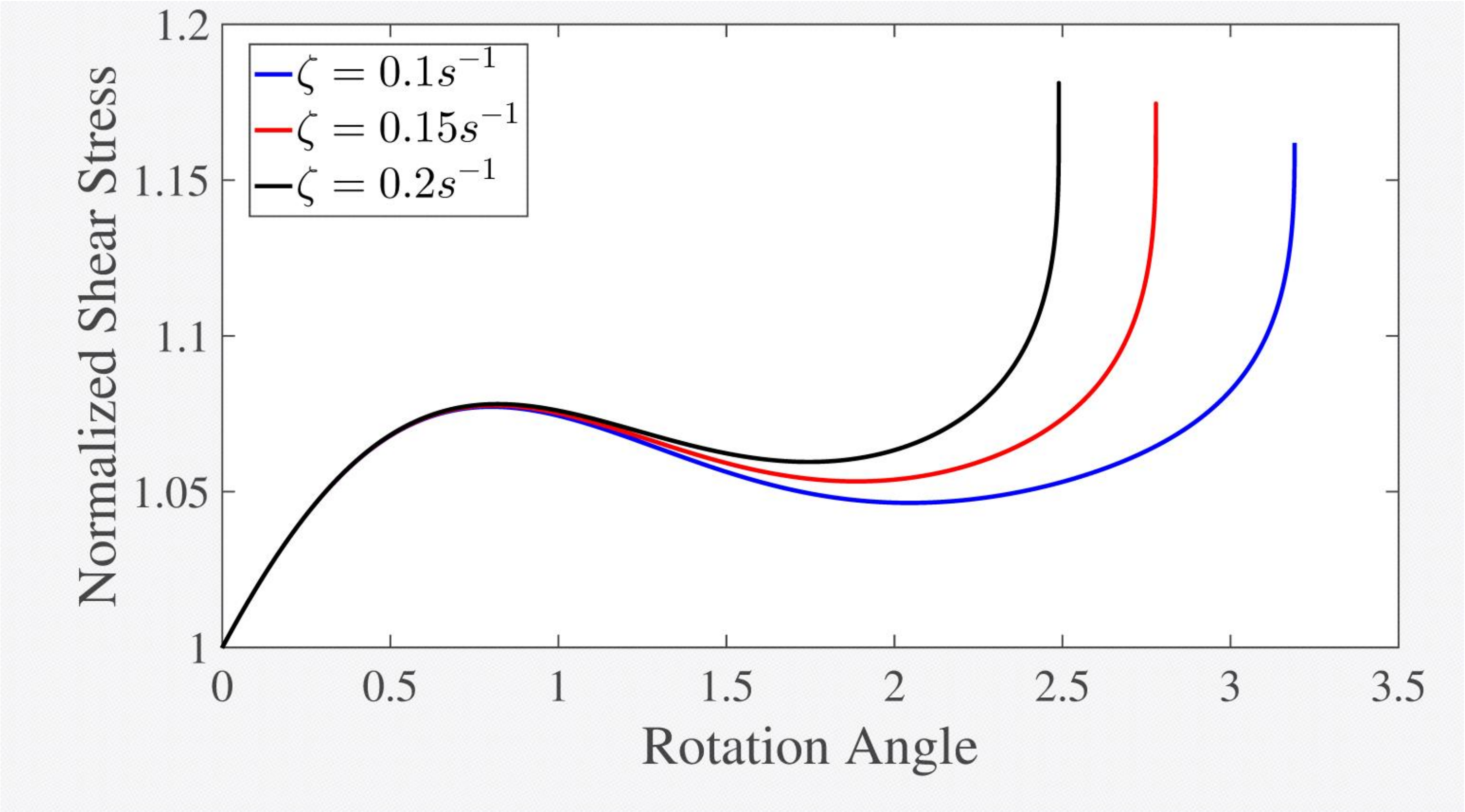
Flow rule

$$\boldsymbol{D}^{\boldsymbol{p}}=\frac{\dot{\gamma}^p}{2\sqrt{J}}\left(\boldsymbol{\sigma}'-\boldsymbol{\alpha}'\right)\Rightarrow\boldsymbol{\sigma}=\boldsymbol{\alpha}'+2\mu\boldsymbol{D}^{\boldsymbol{p}}-p\mathbf{1}\,;\qquad \mu=\tau/\dot{\gamma}^p$$

- ***Rotational Softening***

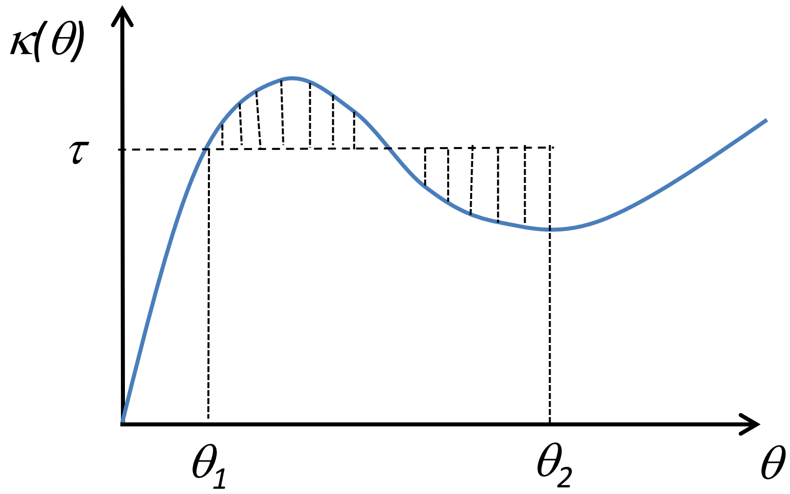
$$\theta = \int \frac{1}{2}\sqrt{\boldsymbol{\omega}\cdot\boldsymbol{\omega}}\; dt = \int \frac{1}{2}\sqrt{\mathrm{tr}(\boldsymbol{\omega}\boldsymbol{\omega}^{T})}\; dt \quad \ldots \text{ rotation angle in 2D}$$

θ… accumulative rotational variable (analogous to plastic strain γ)



• *Inhomogeneous Rotation / Gradient of θ*

$$\theta = \theta(z) \qquad \tau = \kappa(\theta) - a\theta_{zz} - b\theta_z^2$$



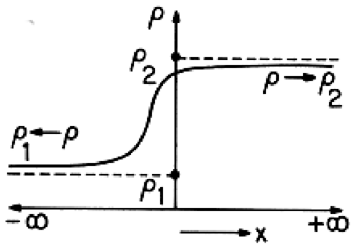
- Analytical Solutions / Conditions for Existence

$$\kappa(\theta_1) = \kappa(\theta_2) = \tau\ , \quad \int_{\theta_1}^{\theta_2} [\kappa(\theta) - \tau] \mathrm{E}(\theta)\, d\theta = 0\ ; \quad \mathrm{E}(\theta) \equiv \frac{1}{a} \exp(2\int \frac{b}{a} d\theta)$$

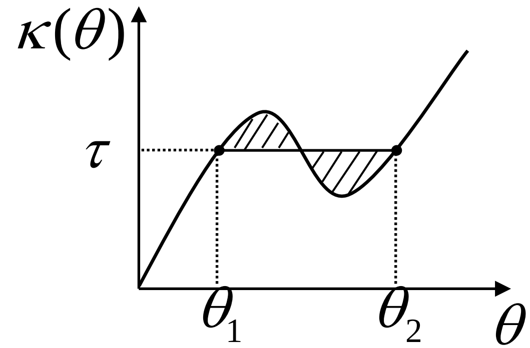
$$z = z_0 + \int_{\theta(z_0)}^{\theta(z)} \frac{d\theta}{\sqrt{2F(\theta)/G(\theta)}}\ ; \quad F \equiv \int_{\theta_1}^{\theta} (\kappa(\theta) - \tau) \mathrm{E}(\theta)\, d\theta\ ; \quad G \equiv \alpha\, \mathrm{E}(\theta)$$

- ***Planar Rotational Bands / 1D Profiles***

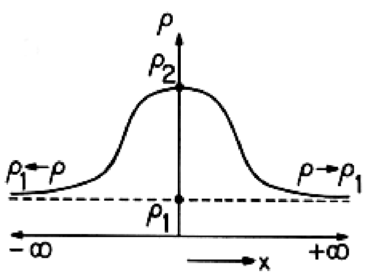
  - *Transitions* $\theta \to \theta_{1,2}$ *as* $z \to \mp\infty$

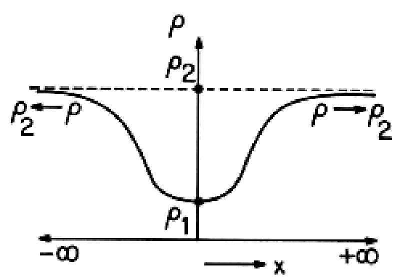


$\rho \equiv \theta$

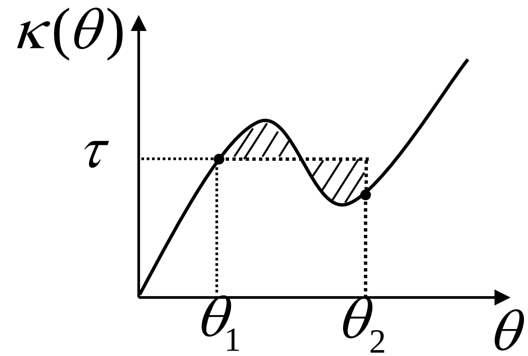


  - *Reversals* $\theta \to \theta_1$ as $z \to \mp\infty$

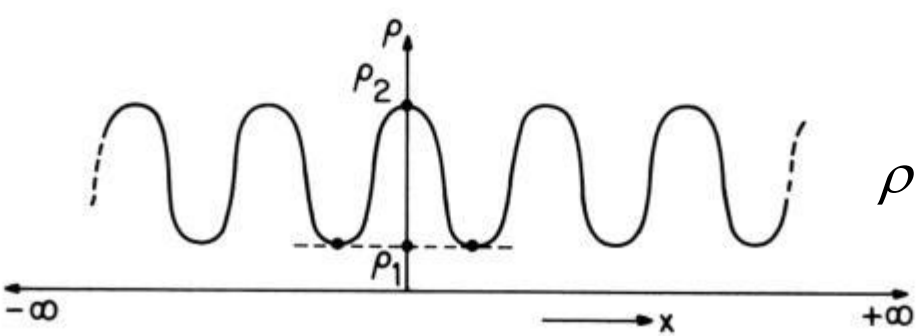




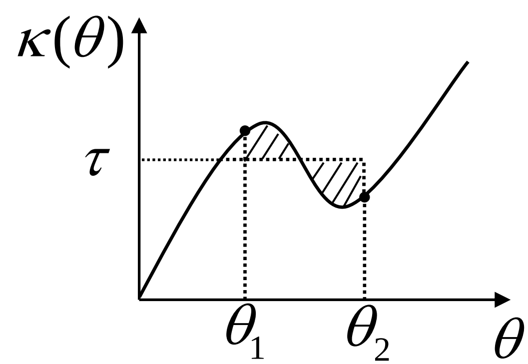
$\rho \equiv \theta$



  - *Oscillations*



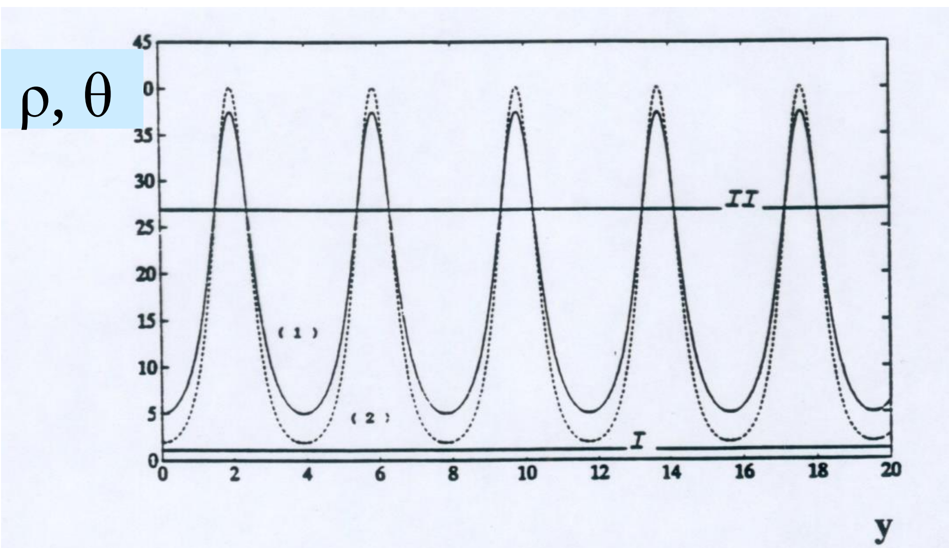
$\rho \equiv \theta$

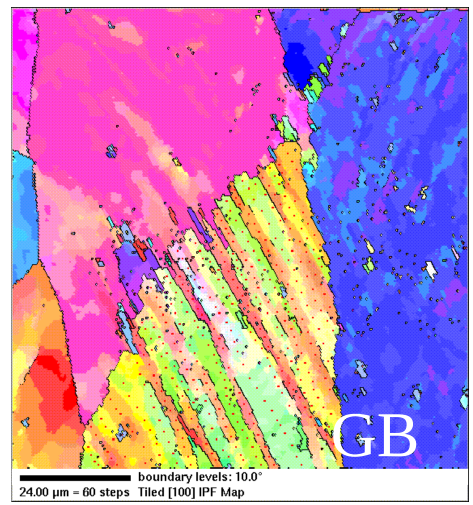


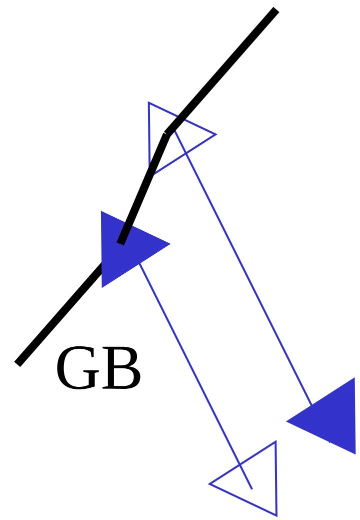
Disclination-dislocation kinetics and patterning

$$\frac{\partial \rho}{\partial t} = F(\rho) - L(\theta)B\rho^2 - M\rho\theta + D\frac{\partial^2 \rho}{\partial x^2}; \qquad \frac{\partial \theta}{\partial t} = -Q(\theta) + \mu M\rho\theta.$$

ρ - density of mobile dislocations; θ - density of immobile disclinations







disclination model

The solution for reaction-kinetic equations
[A.E.Romanov, E.C.Aifantis, Scripta Met. Mat. 29 (1993) 707]

EBSD orientation map image Al, 40% cold rolling [by L. Delannay]

■ Extension to Anisotropic Plastic Flow

- ***Yield condition*** **(Non-Schmid effects)**

$$\tau = \mathrm{tr}\left(\boldsymbol{T}^{L}\boldsymbol{M}\right) = f(\gamma^{p}; \boldsymbol{M}, \boldsymbol{N}, \boldsymbol{A}_{i})$$

- ***Flow rule***

$$\boldsymbol{D}^{p} = \dot{\gamma}^{p}\boldsymbol{M}$$

- ***Determination of M***

$$\max\left\{\mathrm{tr}\left(\boldsymbol{T}^{L}\boldsymbol{D}^{p}\right)\right\}\Big|_{\substack{\mathrm{tr}\,\boldsymbol{M}=0,\ \mathrm{tr}\,\boldsymbol{M}^{2}=1/2 \\ \mathrm{tr}\,\boldsymbol{N}\boldsymbol{M}=0,\ \mathrm{tr}\,\boldsymbol{N}^{2}=1,\ \tau=f}}$$

$$\Rightarrow \qquad \boldsymbol{M} = \frac{1}{2l_{2}}\{\boldsymbol{T}^{L\prime}\dot{\gamma}^{p} - 2l_{5}\boldsymbol{P}\}, \qquad \boldsymbol{P} = \frac{1}{2}\{\boldsymbol{T}^{L\prime} - \boldsymbol{H}_{2}^{\prime} + 2\sqrt{J_{H_{1}}}\boldsymbol{N}^{\prime}\}$$

$$\boldsymbol{H}_{1} = \frac{\partial f}{\partial \boldsymbol{N}}, \qquad \boldsymbol{H}_{2} = \frac{\partial f}{\partial \boldsymbol{M}}, \qquad J_{H_{1}} = \frac{1}{2}\mathrm{tr}(\boldsymbol{H}_{1}^{\prime}\boldsymbol{H}_{1}^{\prime})$$

$$l_{2}, \quad l_{5}: \quad \begin{cases} J\dot{\gamma}^{p} - l_{5}\,\mathrm{tr}(\boldsymbol{P}\boldsymbol{T}^{L\prime}) = l_{2}\,f(\gamma^{p}; \boldsymbol{M}, \boldsymbol{N}, \boldsymbol{A}_{i}) \\ l_{2} = \left\{J(\dot{\gamma}^{p})^{2} - 2l_{5}\dot{\gamma}^{p}\,\mathrm{tr}(\boldsymbol{P}\boldsymbol{T}^{L\prime}) + 2l_{5}\,\mathrm{tr}(\boldsymbol{P}^{2})\right\}^{1/2} \end{cases}$$

■ Classes of Yield Behavior

- ***Vertex type:*** $\tau = \zeta\,\mathrm{tr}(\overset{\circ}{\boldsymbol{T}}{}^{L}\boldsymbol{M}) + \kappa(\gamma^{p})$

$$\boldsymbol{D}^{p} = \frac{\dot{\gamma}^{p}}{2\lambda}\{\boldsymbol{T}^{L\prime} - \bar{\lambda}(\boldsymbol{T}^{L\prime} - \zeta\overset{\circ}{\boldsymbol{T}}{}^{L\prime})\}$$

$$\lambda = \sqrt{(2I_2 J - I_1^2)/(2I_2 - \kappa^2)}\,, \qquad \bar{\lambda} = (I_1 - \lambda\kappa)/2I_2$$

$$I_1 = \mathrm{tr}(\boldsymbol{T}^{L\prime}\boldsymbol{P})\,, \qquad I_2 = \mathrm{tr}(\boldsymbol{P}^2)\,, \qquad \boldsymbol{P} = \frac{1}{2}(\boldsymbol{T}^{L\prime} - \zeta\overset{\circ}{\boldsymbol{T}}{}^{L\prime})$$

$$\dot{\gamma}^{p} = \frac{\overset{\circ}{\boldsymbol{P}}\cdot(\boldsymbol{T}^{L\prime} - 2\bar{\lambda}\boldsymbol{P})}{\lambda\kappa'}$$

- ***Transversely isotropic:*** $\tau = \zeta(\gamma^{p})\,\mathrm{tr}(\boldsymbol{A}\boldsymbol{M}) + \kappa(\gamma^{p})\,, \qquad \boldsymbol{A} = \boldsymbol{a}\otimes\boldsymbol{a}$

$$\boldsymbol{D}^{p} = \frac{\dot{\gamma}^{p}}{2\lambda}\{\boldsymbol{T}^{L\prime} - \bar{\lambda}(\boldsymbol{T}^{L\prime} - \zeta\boldsymbol{A}')\}$$

$$I_1 = \mathrm{tr}(\boldsymbol{T}^{L\prime}\boldsymbol{P})\,, \qquad I_2 = \mathrm{tr}(\boldsymbol{P}^2)\,, \qquad \boldsymbol{P} = \frac{1}{2}(\boldsymbol{T}^{L\prime} - \zeta\boldsymbol{A}')$$

$$\dot{\gamma}^{p} = \left< \frac{(\boldsymbol{T}^{L\prime} - 2\bar{\lambda}\boldsymbol{P})\cdot\overset{\circ}{\boldsymbol{\sigma}}{}'}{(2h + a_m')\lambda + \zeta'\boldsymbol{A}\cdot(\boldsymbol{T}^{L\prime} - 2\bar{\lambda}\boldsymbol{P})} \right>$$

■ Extension to Double Slip

- ***Plastic Stretching***

$$\boldsymbol{D}^{\boldsymbol{p}} = \dot{\gamma}_1^P \boldsymbol{M}_1 + \dot{\gamma}_2^P \boldsymbol{M}_2 \,, \qquad \boldsymbol{M}_i = (\boldsymbol{v}^{(i)} \otimes \boldsymbol{n}^{(i)})_S \,, \qquad (i = 1, 2)$$

Determination of $\boldsymbol{M}_i \quad (\boldsymbol{T}^D = \boldsymbol{\alpha} = \boldsymbol{O})$

$$\max\left\{\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{D}^{\boldsymbol{p}})\right\}\Big|_{\boldsymbol{M}_1}, \quad \mathrm{tr}\,\boldsymbol{M}_1 = 0, \quad \mathrm{tr}\,\boldsymbol{M}_1^2 = \frac{1}{2}$$

$$\max\left\{\mathrm{tr}(\mathring{\boldsymbol{\sigma}}\,\boldsymbol{D}^{\boldsymbol{p}})\right\}\Big|_{\boldsymbol{M}_2}, \quad \mathrm{tr}\,\boldsymbol{M}_2 = 0, \quad \mathrm{tr}\,\boldsymbol{M}_2^2 = \frac{1}{2}$$

$$\Rightarrow \quad \boldsymbol{M}_1 = \frac{1}{2\sqrt{J}}\boldsymbol{\sigma}' \,, \quad \boldsymbol{M}_2 = \frac{1}{2\sqrt{I}}\mathring{\boldsymbol{\sigma}}' \,; \qquad J = \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}'^2) \,, \quad I = \frac{1}{2}\mathrm{tr}(\mathring{\boldsymbol{\sigma}}'^2)$$

$$\therefore \quad \boldsymbol{D}^{\boldsymbol{p}} = \frac{\dot{\gamma}_1^p}{2\sqrt{J}}\boldsymbol{\sigma}' + \frac{\dot{\gamma}_2^p}{2\sqrt{I}}\mathring{\boldsymbol{\sigma}}'$$

- ***Plastic Spin***

$$\boldsymbol{W}^{p} = \dot{\gamma}_1^{P}\boldsymbol{\Omega}_1 + \dot{\gamma}_2^{P}\boldsymbol{\Omega}_2 , \qquad \boldsymbol{\Omega}_i = (\boldsymbol{v}^{(i)} \otimes \boldsymbol{n}^{(i)})_A , \qquad (i = 1, 2)$$

Determination of $\boldsymbol{\Omega}_i$

$$\max\left\{\mathrm{tr}(\mathring{\boldsymbol{\sigma}}\,\boldsymbol{D}^{p})\right\}, \quad \mathrm{tr}\,\boldsymbol{\Omega}_1 = \mathrm{tr}\,\boldsymbol{\Omega}_2 = 0, \quad \mathrm{tr}\,\boldsymbol{\Omega}_1^2 = \mathrm{tr}\,\boldsymbol{\Omega}_2^2 = -\frac{1}{2}$$

$$\Rightarrow \quad \boldsymbol{W}^{p} = -\frac{1}{a_\omega}(\boldsymbol{\sigma}\boldsymbol{D}^{p} - \boldsymbol{D}^{p}\boldsymbol{\sigma}) ; \qquad a_\omega = \frac{1}{\dot{\gamma}^{p}}\sqrt{-2\,\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{D}^{p} - \boldsymbol{D}^{p}\boldsymbol{\sigma})^2}$$

$$\dot{\gamma}^{p} = \dot{\gamma}_1^{p} + \dot{\gamma}_2^{p}$$

- ***Yield Conditions***

$$\tau_i = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_i) = \kappa(\gamma_i^{p})$$

- ***Consistency Conditions*** $(\dot{\tau}_i = 0)$

$$\dot{\gamma}_1^{p} = \frac{\boldsymbol{\sigma}' \cdot \mathring{\boldsymbol{\sigma}}'}{2\kappa\kappa'} \qquad ; \qquad \dot{\gamma}_2^{p} = \frac{2I\ddot{J} - \dot{I}\dot{J}}{2\kappa' I\sqrt{I}}$$

■ Extension to Stress Space Description

$$\boldsymbol{T}^{L'} = 2\tau \boldsymbol{M}\ ,\quad \tau = \sqrt{J} = \sqrt{\tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{T}^{L'}\boldsymbol{T}^{L'})}\ ;\qquad \boldsymbol{T}^{D} = a_m \boldsymbol{M} + a_n \boldsymbol{N}$$

- ***Micro – Micro Transition***

$$\max\left\{\mathrm{tr}\left(\boldsymbol{T}^{L}\boldsymbol{D}^{p}\right)\right\}\Big|_{\mathrm{tr}\,\boldsymbol{M}=0,\ \mathrm{tr}\,\boldsymbol{M}^2=1/2}$$

$$\Rightarrow \dot{\gamma}^{p} = \sqrt{2\boldsymbol{D}^{p}\cdot\boldsymbol{D}^{p}}\ ,\qquad \boldsymbol{M} = \frac{1}{\dot{\gamma}^{p}}\boldsymbol{D}^{p}\ ,\qquad \boldsymbol{D}^{p} = \frac{\dot{\gamma}^{p}}{2\sqrt{J}}\boldsymbol{T}^{L'}$$

Evolution of Back Stress: $\overset{\circ}{\boldsymbol{\alpha}}' = \left(\dfrac{\dot{a}_m a_n - a_m \dot{a}_n}{2\dot{\tau}a_n + \dot{a}_m a_n - a_m \dot{a}_n}\right)\overset{\circ}{\boldsymbol{\sigma}}' + \dfrac{2\dot{\tau}\dot{a}_n}{2\dot{\tau}a_n + \dot{a}_m a_n - a_m \dot{a}_n}\boldsymbol{\alpha}'$

$$\boldsymbol{W}^{p} = -\frac{1}{a_n}\left(\boldsymbol{\sigma}'\boldsymbol{D}^{p} - \boldsymbol{D}^{p}\boldsymbol{\sigma}'\right)$$

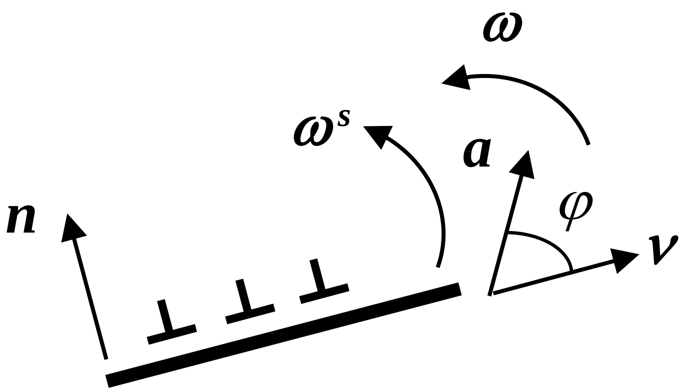
M1: $\dot{a}_m = \dfrac{2\mu_1}{1-\mu_1}\dot{\tau},\quad a_n = const. \quad\Rightarrow\quad \overset{\circ}{\boldsymbol{\alpha}}' = \mu_1 \overset{\circ}{\boldsymbol{\sigma}}'$ … Phillips' rule

M2: $\dot{a}_m = \dfrac{2\mu_1}{1-\mu_1}\dot{\tau} - \dfrac{\mu_2\dot{\bar{\varepsilon}}}{1-\mu_1}a_m,\quad \dot{a}_n = -\dfrac{\mu_2\dot{\bar{\varepsilon}}}{1-\mu_1}a_n \quad\Rightarrow\quad \overset{\circ}{\boldsymbol{\alpha}}' = \mu_1 \overset{\circ}{\boldsymbol{\sigma}}' - \mu_2\dot{\bar{\varepsilon}}\boldsymbol{\alpha}'$

… Extension of Phillips' rule

■ Extension to Textured Polycrystals

- **Texture vector**



$$\boldsymbol{a} = (\cos\varphi)\boldsymbol{v} + (\sin\varphi)\boldsymbol{n}$$

$$\dot{\boldsymbol{a}} = \boldsymbol{\omega}\boldsymbol{a}, \quad \dot{\boldsymbol{n}} = \boldsymbol{\omega}_s\boldsymbol{n}, \quad \dot{\boldsymbol{v}} = \boldsymbol{\omega}_s\boldsymbol{v}$$

$$\boldsymbol{M} = (\boldsymbol{v}\otimes\boldsymbol{n})_s = \frac{1}{2}(\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{a}\otimes\boldsymbol{a})\sin 2\varphi + (\boldsymbol{a}\otimes\boldsymbol{b})_s\cos 2\varphi$$

$$\boldsymbol{\Omega} = (\boldsymbol{v}\otimes\boldsymbol{n})_a = (\boldsymbol{a}\otimes\boldsymbol{b})_a$$

$$\boldsymbol{b} = (-\sin\varphi)\boldsymbol{v} + (\cos\varphi)\boldsymbol{n}$$

$$\Rightarrow \quad \boldsymbol{\omega} = \boldsymbol{\omega}^s - \boldsymbol{W}^t; \qquad \boldsymbol{W}^t = 2\dot{\varphi}\boldsymbol{\Omega}$$

- **Kinematics**

$$\boldsymbol{D}^{\boldsymbol{p}} = \dot{\gamma}^{p} \boldsymbol{M}, \quad \boldsymbol{W}^{\boldsymbol{p}} = \dot{\gamma}^{p} \boldsymbol{\Omega}, \quad \boldsymbol{T}^{\boldsymbol{D}} = t_m \boldsymbol{M} + t_n \boldsymbol{N} \quad \text{...as before}$$

But now, two independent spins for polycrystalline aggregate: $\boldsymbol{\omega}^{s}, \boldsymbol{\omega}$

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{R}_t \boldsymbol{F}^{\boldsymbol{p}}, \qquad \boldsymbol{\omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T, \qquad \boldsymbol{W}^{\boldsymbol{t}} = \boldsymbol{R}\dot{\boldsymbol{R}}_t \boldsymbol{R}_t^T \boldsymbol{R}^T$$

$$\boldsymbol{W}^{\boldsymbol{p}} = \left(\boldsymbol{R}\boldsymbol{R}_t \dot{\boldsymbol{F}}^{\boldsymbol{p}} \boldsymbol{F}^{\boldsymbol{p}\,-1} \boldsymbol{R}_t^T \boldsymbol{R}^T \right)_a, \qquad \boldsymbol{D}^{\boldsymbol{p}} = \left(\boldsymbol{R}\boldsymbol{R}_t \dot{\boldsymbol{F}}^{\boldsymbol{p}} \boldsymbol{F}^{\boldsymbol{p}\,-1} \boldsymbol{R}_t^T \boldsymbol{R}^T \right)_s$$

$$\therefore \quad \boldsymbol{\omega} = \underbrace{\boldsymbol{W} - \boldsymbol{W}^{\boldsymbol{p}}}_{\boldsymbol{\omega}^{s}} - \boldsymbol{W}^{\boldsymbol{t}}$$

$$\boldsymbol{W}^{\boldsymbol{p}} = \lambda (\boldsymbol{A}\boldsymbol{D}^{\boldsymbol{p}} - \boldsymbol{D}^{\boldsymbol{p}}\boldsymbol{A}), \qquad \boldsymbol{W}^{t} = 2\lambda \frac{d\varphi}{d\gamma^{p}} (\boldsymbol{A}\boldsymbol{D}^{\boldsymbol{p}} - \boldsymbol{D}^{\boldsymbol{p}}\boldsymbol{A});$$

$$\lambda = \sec 2\varphi, \qquad \boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a}$$

• Geometry of Polycrystalline Aggregate

Orientation Distribution Function (ODF): The probability of a grain being oriented along $\boldsymbol{a}$ at time t is defined by ODF …. $\psi(\boldsymbol{a},t)$

Conservation Law

$$\frac{d}{dt}\oint \psi(\boldsymbol{a},t)\,d\boldsymbol{a} = 0 \qquad \Rightarrow \qquad \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial a_i}(\dot{a}_i \psi) = 0$$

- $\psi(\theta,\phi;\,t) = \psi(\pi+\theta,\pi-\phi;\,t)\,, \qquad \oint \psi(\boldsymbol{a},t)\,d\boldsymbol{a} = 1$
- $\psi(\theta;0) = 1/\pi \ldots(2\text{D}), \qquad \psi(\theta,\phi;0) = 1/4\pi \ldots(3\text{D})$

General Solution (Dihn and Armstrong, 1984)

$$\dot{\boldsymbol{a}} = (\boldsymbol{LA} - \boldsymbol{AL})\boldsymbol{a} \quad \rightarrow \quad \boldsymbol{a} = \frac{\boldsymbol{F}\boldsymbol{a}_0}{\left(\boldsymbol{F}^T\boldsymbol{F}\cdot\boldsymbol{a}_0\otimes\boldsymbol{a}_0\right)^{1/2}}$$

$$\Rightarrow \begin{cases} \psi(\boldsymbol{a},t) = \dfrac{1}{\pi}\left[\boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\cdot\boldsymbol{A}\right]^{-1} & \ldots(2\text{D}) \\ \psi(\boldsymbol{a},t) = \dfrac{1}{4\pi}\left[\boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\cdot\boldsymbol{A}\right]^{-3/2} & \ldots(3\text{D}) \end{cases}$$

- **Geometry of Continuum (*Oriented Continuum*)**

Fourier series (2D) or spherical harmonics (3D) expansion (*Gelfand*, *et al.*, *1963*)

$$\psi(\boldsymbol{a},t)=\frac{1}{4\pi}\left(1+b_2B_{ij}a_ia_j+b_4C_{ijkl}a_ia_ja_ka_l+\ldots\right)$$

$$b_2=\frac{5}{4}\begin{pmatrix}4\\2\end{pmatrix},\qquad b_4=\frac{9}{16}\begin{pmatrix}8\\4\end{pmatrix}$$

$B_{ij}=\oint a_ia_j\psi(\boldsymbol{a},t)\,d\boldsymbol{a}$ ...2nd order orientation tensor (2nd moment)

$C_{ijkl}=\oint a_ia_ja_ka_l\psi(\boldsymbol{a},t)\,d\boldsymbol{a}$ ...4th order orientation tensor (4th moment)

• Micro – Macro Transition

This involves two successive steps

- *First step: single slip $\rightarrow$ crystallite level*

Scale invariance argument and maximization procedure … as before

$$\boldsymbol{D}^{p} = \frac{\dot{\gamma}^{p}}{2\kappa(\gamma^{p})}\boldsymbol{T}^{L'} \; ; \qquad \overset{\circ}{\boldsymbol{T}}{}^{D} = \left(\frac{\dot{t}_m}{\dot{\gamma}^{p}} - \frac{\dot{t}_n t_m}{t_n \dot{\gamma}^{p}} \right) \boldsymbol{D}^{p} + \frac{\dot{t}_n}{t_n} \boldsymbol{T}^{D}$$

$$\overset{\circ}{\boldsymbol{T}}{}^{D} = \dot{\boldsymbol{T}}^{D} - \boldsymbol{\omega}^{s}\boldsymbol{T}^{D} + \boldsymbol{T}^{D}\boldsymbol{\omega}^{s} \; ; \qquad \boldsymbol{\omega}^{s} = \boldsymbol{W} - \boldsymbol{W}^{p}$$

$$\boldsymbol{W}^{p} = -\frac{1}{t_n}\left(\boldsymbol{T}^{D}\boldsymbol{D}^{p} - \boldsymbol{D}^{p}\boldsymbol{T}^{D} \right)$$

- *Second step: crystallite $\rightarrow$ macroscopic level*

(Taylor’s argument and the following average procedure)

❑ Average Stress

$$\boldsymbol{\sigma}' = \oint \boldsymbol{K}\,\boldsymbol{S}'\psi(\boldsymbol{a},t)\,d\boldsymbol{a}\,; \qquad \boldsymbol{\alpha}' = \oint \boldsymbol{K}\,\boldsymbol{T}^{D'}\psi(\boldsymbol{a},t)\,d\boldsymbol{a}$$

$\boldsymbol{K}$ …4th order texture tensor: $K_{ijkl} = K_{jikl} = K_{ijlk} = K_{klij}$

$$K_{ijkl} = k_1 a_i a_j a_k a_l + k_2(a_i a_j \delta_{kl} + a_k a_l \delta_{ij}) + k_3(a_i a_k \delta_{jl} + a_i a_l \delta_{jk} + a_j a_k \delta_{il} + a_j a_l \delta_{ik}) + k_4 \delta_{ij}\delta_{kl} + k_5(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Incompressibility: $K_{ijkk} = 0 \quad \Rightarrow$

$$k_1 + 2k_2 + 4k_3 = 0\,, \qquad k_2 + 2k_4 + 2k_5 = 0 \quad \ldots(2D)$$

$$k_1 + 3k_2 + 4k_3 = 0\,, \qquad k_2 + 3k_4 + 2k_5 = 0 \quad \ldots(3D)$$

Note: $k_1 = k_2 = 0,\ k_5 = 1/2 \quad \Rightarrow \quad \boldsymbol{\sigma}' = \oint \boldsymbol{S}'\psi(\boldsymbol{a},t)\,d\boldsymbol{a}$

i.e. conventional “averaging” formula

❑ Flow Rule

Average Procedure and Taylor's Assumption ($\boldsymbol{D}^p = \bar{\boldsymbol{D}}^p$) $\Rightarrow$

$$\bar{\boldsymbol{D}}^p = \frac{\dot{\bar{\gamma}}^p}{2\kappa(\bar{\gamma}^p)}\langle \boldsymbol{K}\rangle^{-1}(\boldsymbol{\sigma}' - \boldsymbol{\alpha}')$$

$$\langle K\rangle_{ijkl} = k_1 C_{ijkl} + k_2(B_{ij}\delta_{kl} + B_{kl}\delta_{ij}) + k_3(B_{ik}\delta_{jl} + B_{il}\delta_{jk} + B_{jk}\delta_{il} + B_{jl}\delta_{ik}) + k_4\delta_{ij}\delta_{kl} + k_5(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

❑ Evolution Equation of Back Stress

Average Procedure and Taylor's Assumption ($\boldsymbol{D}^p = \bar{\boldsymbol{D}}^p$) $\Rightarrow$

$$\overset{\circ}{\boldsymbol{\alpha}}' = \left(\frac{\dot{t}_m}{\dot{\bar{\gamma}}^p} - \frac{\dot{t}_n t_m}{t_n \dot{\bar{\gamma}}^p}\right)\langle \boldsymbol{K}\rangle \bar{\boldsymbol{D}}^p + \frac{\dot{t}_n}{t_n}\boldsymbol{\alpha}'$$

$$\overset{\circ}{\boldsymbol{\alpha}}' = \dot{\boldsymbol{\alpha}} - \bar{\boldsymbol{\varpi}}^s\boldsymbol{\alpha} + \boldsymbol{\alpha}\bar{\boldsymbol{\varpi}}^s\ ; \qquad \bar{\boldsymbol{\varpi}}^s = \boldsymbol{W} - \bar{\boldsymbol{W}}^p, \qquad \bar{\boldsymbol{W}}^p = -\frac{1}{t_n}\left(\boldsymbol{\alpha}'\bar{\boldsymbol{D}}^p - \bar{\boldsymbol{D}}^p\boldsymbol{\alpha}'\right)$$

Note: Scale invariance argument and Taylor's assumption was used to obtain $\bar{\boldsymbol{W}}^p$

❑ Yield Condition

$$\left\{ \frac{1}{2} \mathrm{tr}\left(\langle \boldsymbol{K} \rangle^{-1} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}') \right)^2 \right\}^{1/2} = \kappa(\bar{\gamma}^p)$$

❑ Consistency Condition

$$\dot{\bar{\gamma}}^p = \frac{1}{2\kappa'\kappa} \{ \overline{\langle \boldsymbol{K} \rangle^{-1} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}')}^{\square} \cdot \langle \boldsymbol{K} \rangle^{-1} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}') \}$$

$$\overset{\square}{\boldsymbol{\sigma}}{}' = \dot{\boldsymbol{\sigma}}' - \bar{\boldsymbol{\omega}} \boldsymbol{\sigma}' + \boldsymbol{\sigma}' \bar{\boldsymbol{\omega}} \, ; \qquad \bar{\boldsymbol{\omega}} = \boldsymbol{W} - \bar{\boldsymbol{W}}^{\boldsymbol{p}} - \bar{\boldsymbol{W}}^{\boldsymbol{t}}$$

❑ Texture Spin

$$\bar{\boldsymbol{w}}^{\boldsymbol{t}} = \oint \boldsymbol{\Lambda} \boldsymbol{w}^{\boldsymbol{t}} \psi(\boldsymbol{a}, t) \, d\boldsymbol{a} \, ; \qquad \bar{W}_{ij}^{t} = -\varepsilon_{ijk} \bar{w}_k^t$$

2nd order texture tensor: $\Lambda_{ij} = \Lambda_1 a_i a_j + \Lambda_2 \delta_{ij} \quad \Rightarrow \quad (\bar{\lambda} = \lambda \Lambda_2)$

$$\therefore \quad \bar{\boldsymbol{W}}^{\boldsymbol{t}} = \bar{\lambda} (\boldsymbol{B} \bar{\boldsymbol{D}}^{\boldsymbol{p}} - \bar{\boldsymbol{D}}^{\boldsymbol{p}} \boldsymbol{B})$$

• *Structure of Texture Plasticity*

$$\boldsymbol{D}^p = \frac{\dot{\gamma}^p}{2\kappa(\gamma^p)} \langle \boldsymbol{K} \rangle^{-1} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}')$$

$$\langle K \rangle_{ijkl} = k_1 C_{ijkl} + k_2 (B_{ij}\delta_{kl} + B_{kl}\delta_{ij}) + k_3 (B_{ik}\delta_{jl} + B_{il}\delta_{jk} + B_{jk}\delta_{il} + B_{jl}\delta_{ik}) + k_4 \delta_{ij}\delta_{kl} + k_5 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\mathring{\boldsymbol{\alpha}}' = \left(\frac{\dot{t}_m}{\dot{\gamma}^p} - \frac{\dot{t}_n t_m}{t_n \dot{\gamma}^p} \right) \langle \boldsymbol{K} \rangle \boldsymbol{D}^p + \frac{\dot{t}_n}{t_n} \boldsymbol{\alpha}', \qquad \boldsymbol{W}^p = -\frac{1}{t_n} \left(\boldsymbol{\alpha}' \boldsymbol{D}^p - \boldsymbol{D}^p \boldsymbol{\alpha}' \right)$$

$$\mathring{\boldsymbol{\alpha}}' = \dot{\boldsymbol{\alpha}} - \boldsymbol{\omega}^s \boldsymbol{\alpha} + \boldsymbol{\alpha}\boldsymbol{\omega}^s ; \qquad \boldsymbol{\omega}^s = \boldsymbol{W} - \boldsymbol{W}^p$$

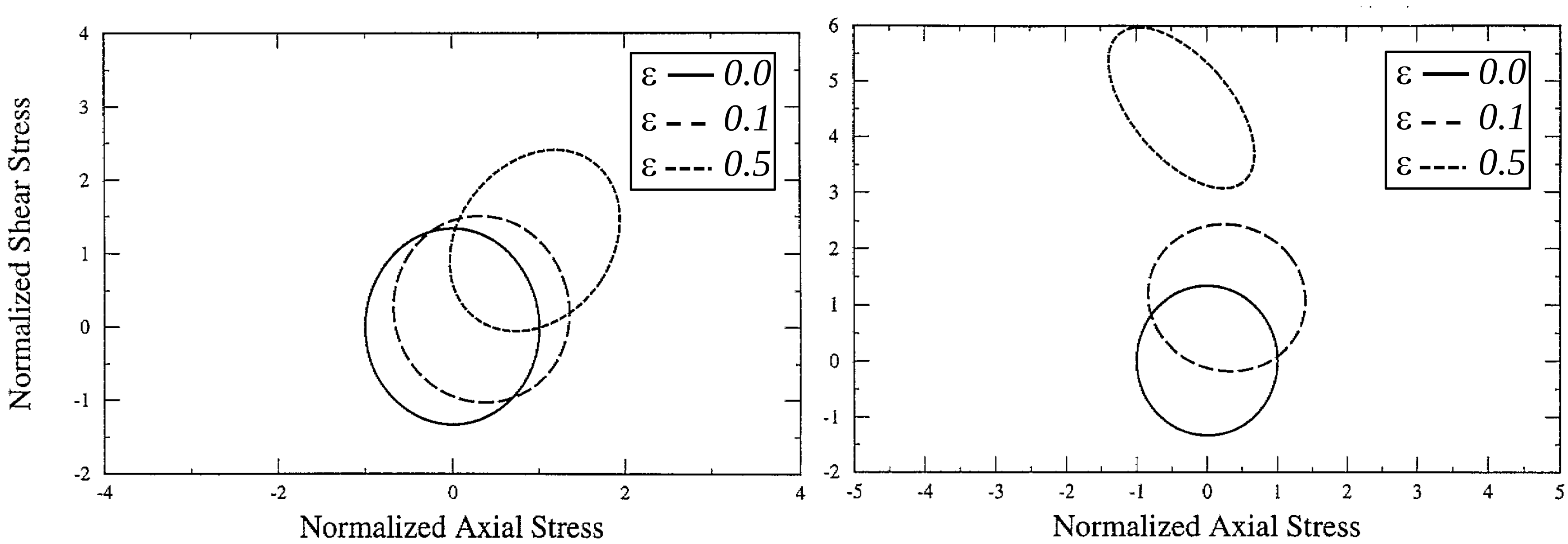
$$\left\{ \frac{1}{2} \mathrm{tr}(\langle \boldsymbol{K} \rangle^{-1} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}'))^2 \right\}^{1/2} = \kappa(\bar{\gamma}^p),$$

$$\dot{\bar{\gamma}}^p = \frac{1}{2\kappa'\kappa} \{ \overline{\langle \boldsymbol{K} \rangle^{-1} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}')}^{\square} \cdot \langle \boldsymbol{K} \rangle^{-1} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}') \}$$

$$\overset{\square}{\boldsymbol{\sigma}}' = \dot{\boldsymbol{\sigma}}' - \bar{\boldsymbol{\omega}}\boldsymbol{\sigma}' + \boldsymbol{\sigma}'\bar{\boldsymbol{\omega}} ; \qquad \bar{\boldsymbol{\omega}} = \boldsymbol{W} - \bar{\boldsymbol{W}}^p - \bar{\boldsymbol{W}}^t ; \qquad \boldsymbol{W}^t = \bar{\lambda} (\boldsymbol{B}\boldsymbol{D}^p - \boldsymbol{D}^p \boldsymbol{B})$$

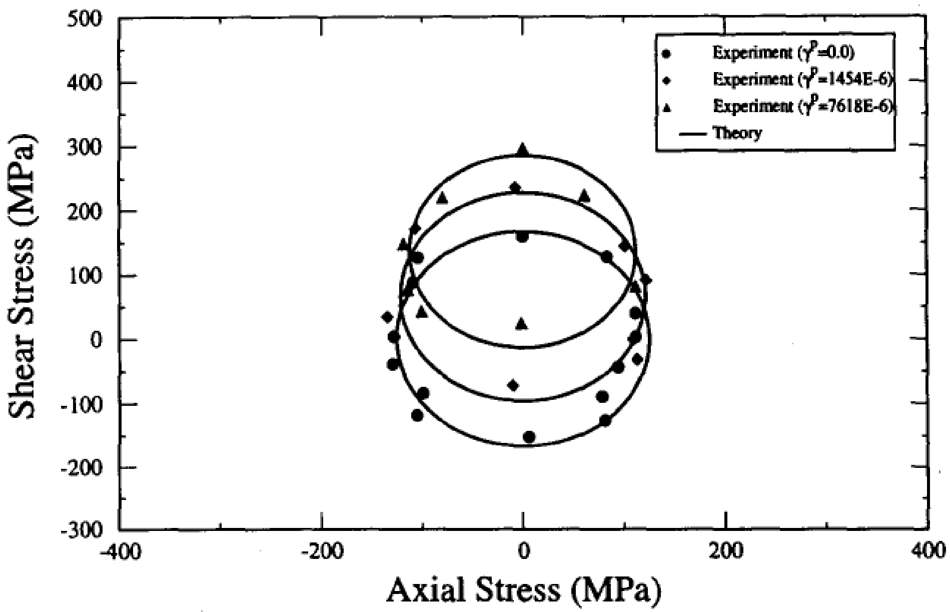
• *Tension – Torsion Example*

$$[\boldsymbol{D}] = \frac{1}{2}\begin{bmatrix} -\dot{\varepsilon} & \dot{\gamma} & 0 \\ \dot{\gamma} & 2\dot{\varepsilon} & 0 \\ 0 & 0 & -\dot{\varepsilon} \end{bmatrix}, \qquad [\boldsymbol{W}] = \frac{\dot{\gamma}}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

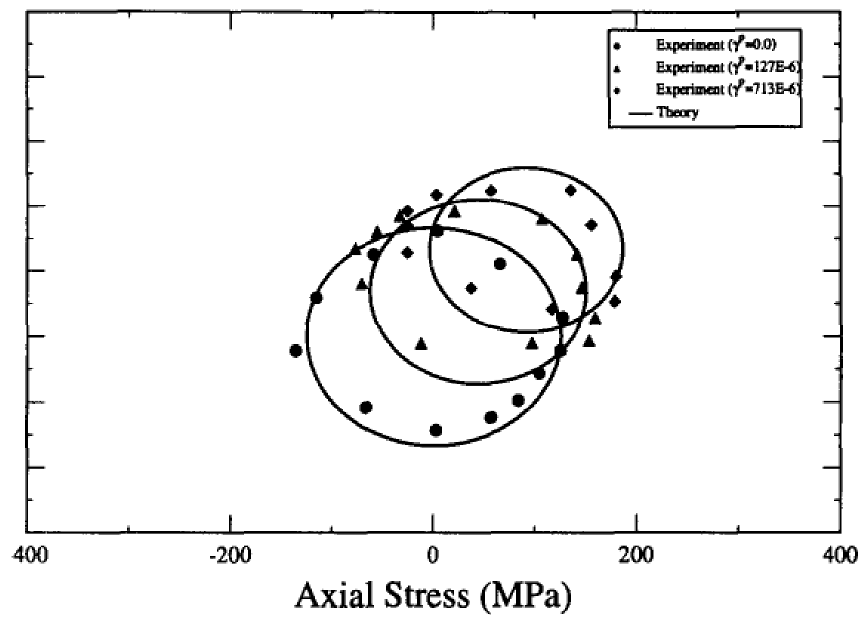


Evolution of yield surface in tension-torsion deformation for $\dfrac{\dot{\gamma}}{\dot{\varepsilon}} = 1$ and 5

• *Comparison with Experimental Data*



Evolution of yield surface of
304 stainless steel in torsion



Evolution of yield surface of
304 stainless steel in tension-torsion